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A Laplace operator with boundary conditions singular at one point

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Abstract
We discuss a recent paper of Berry and Dennis (*J. Phys. A: Math. Theor.* 2008 **41** 135203) concerning a Laplace operator on a smooth domain with singular boundary condition. We explain a paradox in the article (*J. Phys. A: Math. Theor.* 2008 **41** 135203) and show that if a certain additional condition is imposed, the result is a spectral problem for a self-adjoint operator having only eigenvalues and no continuous spectrum. The eigenvalues accumulate at $\pm\infty$ only, and we obtain the asymptotic behaviours of the counting functions $n_+(\lambda)$ and $n_-(\lambda)$ for positive and negative eigenvalues. The physical meaning of the additional boundary condition is not yet clear.

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1. Introduction

Recently Berry and Dennis [3] considered a spectral problem

$$-\Delta u = \lambda u \tag{1}$$

for the Laplacian in a circular domain, with boundary conditions of the form

$$u + f(x)\frac{\partial u}{\partial \nu} = 0.$$

Here $\partial/\partial\nu$ denotes outward normal derivative. The smooth real function f has a finite number of simple zeros and simple poles on the boundary. Writing $\partial u/\partial\nu = (f(x))^{-1}u$, the singularities at the zeros of f (called Dirichlet singularities in [3]) reflect themselves in the properties of the spectrum of the problem.

A paradoxical result was obtained in [3]: it was found that the whole real line is filled by the spectrum; moreover all numbers are eigenvalues, with corresponding eigenfunctions belonging to L_2. This observation contradicts general theorems about the eigenfunctions of self-adjoint operators (see, e.g. [12]): eigenfunctions of a self-adjoint operator, corresponding to different eigenvalues, are orthogonal; thus, since the Hilbert space L_2 is separable, there can be no more than a countable set of eigenvalues, in no way the whole real line.

In the present paper, we try to explain the reason for this strange form of the spectrum and propose a more adequate problem setting. A detailed explanation of the paradox in [3] is given. We show that in the case of a special geometry the problem has only a discrete spectrum with eigenvalues accumulating only at $\pm\infty$ and not at any finite point:

$$\cdots < \lambda_{-2} < \lambda_{-1} < \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots;$$
$$\lambda_k \to \pm\infty \qquad \text{as} \quad k \to \pm\infty.$$

The techniques used are classical and require separation of variables and some knowledge of the Titchmarsh–Weyl limit-point—limit-circle classification of singular Sturm–Liouville problems for ODEs. The asymptotics of the counting function of the eigenvalues are found; in particular, it turns out that the negative eigenvalues are placed more sparsely than the positive ones.

We also give an outline of how the qualitative result can be proved for more general geometries using a domain decomposition trick usually named after Glazman, who proposed it for ordinary differential equations. We should mention, however, that a physical model giving rise to the half-plane problem was already studied and analysed mathematically by Exner and Šeba [5] in 1988.

2. Symmetric and self-adjoint operators

We recall here some facts about unbounded operators in a Hilbert space; for details see [12] or other standard sources.

Let H be a Hilbert space with scalar product $(\cdot,\cdot)$ and let A_0 be an unbounded linear operator in H defined on the domain $D_0 = D(A_0)$, dense in H. The operator is called *symmetric* if $(A_0u, v) = (u, A_0v)$ for all $u, v \in D_0$. If T is any linear operator with dense domain $D(T)$, the *adjoint* of T can be defined, see [12, vol 1, p 252]; the adjoint operator is denoted by T^* and its domain by $D(T^*)$. The symmetric operator A_0 is *self-adjoint* if $A_0^* = A_0$. Any self-adjoint operator is symmetric but the converse is false. To prove that A_0 is in fact self-adjoint one needs to check that the domain of A_0^* coincides with the domain of A_0, and this task might not be that easy.

So, having a symmetric operator A_0, one usually needs to choose a *self-adjoint extension* of A_0, i.e., a self-adjoint operator A such that $D(A) \supset D_0$ and $Au = A_0u$ for $u \in D_0$. Such an extension, if exists, must, in turn, be a *restriction* of the operator A_0^*, so, $D(A) \subset D(A_0^*)$ and $Au = A_0^*u$ for $u \in D(A)$. In such cases the term *self-adjoint realization* is also used.

Why take trouble with this notion of self-adjointness? The reason is that only self-adjoint realizations generate dynamics. If an operator A is only symmetric but not self-adjoint, thus being defined on too small a set, the Schrödinger equation $-\mathrm{i}\frac{\mathrm{d}\psi}{\mathrm{d}t} = A\psi$ does not define a unitary evolution group. An extensive discussion can be found, e.g., in [12, vol 2]. The same happens in the opposite case, when the operator is defined on too large a domain, thus losing symmetry. In both cases the study of the spectrum is useless: the spectrum covers the whole complex plane.

For differential operators in a region in $\mathbb{R}^d$, usually, the domain of the operator is determined by the smoothness of functions inside the region (e.g., $u \in H^2_{\mathrm{loc}}$ for the Laplace operator) and by boundary conditions (at the boundary of the region or at infinity). If one imposes too many boundary conditions, say, simultaneously Dirichlet and Neumann ones for the Laplacian, the operator will be symmetric but not self-adjoint. If one imposes too few boundary conditions or even no conditions at all, the operator ceases to be symmetric.

In regular situations, the correct amount of boundary conditions to impose is usually clear. However in the presence of various kinds of singularities the question about the correct statement of boundary conditions may become rather intricate. In the following section we consider the Berry–Dennis model in more detail and resolve the question of self-adjointness.

3. Problem statement and solution: separable geometry

We consider the Laplacian in the semi-disc in $\mathbb{R}^2$ described in plane polar coordinates by

$$\Omega_1 = \{(r\cos\theta, r\sin\theta) | 0 < r < 1, -\pi/2 < \theta < \pi/2\}.$$

Taking $\epsilon > 0$, we equip the Laplacian in $L^2(\Omega_1)$ with the Dirichlet boundary conditions

$$u(x, y) = 0 \qquad \text{if} \quad x^2 + y^2 = 1, \tag{2}$$

on the semi-circular component

$$\Gamma = \{(\cos\theta, \sin\theta) | -\pi/2 \leqslant \theta \leqslant \pi/2\}$$

of the boundary of Ω_1, and with the Robin-type condition

$$u + \epsilon y \frac{\partial u}{\partial \nu} = 0, \qquad \text{for} \quad x = 0, \quad -1 < y < 1, \tag{3}$$

on the remainder of the boundary. In polar coordinates the boundary condition (3) becomes

$$u + \epsilon \frac{\partial u}{\partial \theta} = 0 \qquad \text{for} \quad 0 < r < 1, \quad \theta = \pm\pi/2. \tag{4}$$

Let L denote the negative Laplacian equipped with the boundary conditions (2), (3): more precisely, the domain of L is

$$D(L) = \{u \in L^2(\Omega_1) | \Delta u \in L^2(\Omega_1) \text{ and } u \text{ satisfies (2, 3) in the weak sense}\} \tag{5}$$

and

$$Lu = -\Delta u := -u_{xx} - u_{yy}. \tag{6}$$

Here all derivatives and boundary conditions should be understood in the usual weak sense.

Now consider the regular self-adjoint Sturm–Liouville problem

$$-\Theta''(\theta) = \lambda\Theta(\theta), \qquad -\pi/2 < \theta < \pi/2; \tag{7}$$

$$\Theta(-\pi/2) + \epsilon\Theta'(-\pi/2) = 0 = \Theta(\pi/2) + \epsilon\Theta'(\pi/2). \tag{8}$$

Let $\mu_n, \Theta_n, n = 0, 1, 2, \ldots,$ denote the eigenvalues and eigenfunctions of this problem. A simple calculation shows that

$$\mu_0 = \frac{-1}{\epsilon}; \qquad \mu_n = n^2, \qquad n = 1, 2, \ldots, \tag{9}$$

$$\Theta_n(\theta) = \begin{cases} \cos(n\theta) - (n\epsilon)^{-1}\sin(n\theta) & (n \text{ even}); \\ \cos(n\theta) + n\epsilon\sin(n\theta) & (n \text{ odd}). \end{cases} \tag{10}$$

A standard calculation (separation of variables: writing $u(x, y) = \sum_n U_n(r)\Theta_n(\theta)$) now shows that the operator L admits orthogonal decomposition into a direct sum of ordinary differential operators:

$$L = \bigoplus_{n=0}^{\infty} L_n, \tag{11}$$

where

$$D(L_n) = \left\{ U \in L^2_r(0,1) \middle| -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}U}{\mathrm{d}r}\right) + \mu_n r^{-2} U \in L^2_r(0,1); U(1) = 0 \right\}, \tag{12}$$

$$L_n U = -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}U}{\mathrm{d}r}\right) + \mu_n r^{-2} U. \tag{13}$$

Here $L^2_r(0,1)$ denotes the space of functions U which satisfy the weighted square integrability condition

$$\int_0^1 r|U(r)|^2\,\mathrm{d}r < +\infty. \tag{14}$$

The operators L_n for $n \geqslant 1$ differ from the operator L_0 in one important respect: L_0 is not self-adjoint, but the L_n for $n \geqslant 1$ are all self-adjoint, for reasons which we now explain.

The singularity at the origin for the eigenvalue problems

$$-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}U}{\mathrm{d}r}\right) + \frac{n^2}{r^2}U = \lambda U, \tag{15}$$

$$U(1) = 0 \tag{16}$$

is of the so-called limit-point type (see Reed and Simon [12, vol 2, p 152], Coddington and Levinson [4, chapter 9]): in other words, for each λ the differential equation

$$-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}U}{\mathrm{d}r}\right) + \frac{n^2}{r^2}U = \lambda U$$

has at most a one-dimensional space of solutions satisfying the square integrability condition (14). In fact, for each n, the space of square integrable solutions has dimension precisely 1 and is spanned by the Bessel function $J_n(r\sqrt{\lambda})$. All other solutions have the form $AJ_n(r\sqrt{\lambda}) + BY_n(r\sqrt{\lambda})$, where $B \neq 0$, and are therefore not square integrable.

The square integrability condition can be regarded as acting like a second-boundary condition, replacing the condition which one would ordinarily impose at $r = 0$ if this were a regular endpoint. It can be shown [4, chapter 9, problem 13] that, subject to square integrability, the L_n do not require a further boundary condition at the singular point. The square integrability condition selects $U(r) = \text{const}\, J_n(r\sqrt{\lambda})$ as the only admissible solution and the remaining boundary condition $U(1) = 0$ then restricts $\sqrt{\lambda}$ to be a zero of J_n. The eigenfunctions of each L_n thus have the form

$$R_{n,k}(r) = J_n(rj_{n,k}), \tag{17}$$

where $j_{n,k}$ denotes the kth zero of the Bessel function J_n, and the corresponding eigenvalues are $\lambda_{n,k} = j^2_{n,k}$.

The operator L_0, by contrast, is of the so-called limit-circle type [12, p 151], [4, chapter 9]: in other words, for each λ, all solutions of the differential equation

$$L_0 U = -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}U}{\mathrm{d}r}\right) - \frac{1}{\epsilon^2 r^2}U = \lambda U \tag{18}$$

satisfy the square integrability condition (14), just as if $r = 0$ were a regular endpoint. Given any $\lambda \in \mathbb{C}$, the function

$$U(r) = J_{i/\epsilon}(r\sqrt{\lambda})Y_{i/\epsilon}(\sqrt{\lambda}) - Y_{i/\epsilon}(r\sqrt{\lambda})J_{i/\epsilon}(\sqrt{\lambda})$$

is a nontrivial solution of the differential equation which satisfies the boundary condition $U(1) = 0$ together with the square integrability condition (14): thus, any complex number

λ is an eigenvalue of L_0. Since a self-adjoint operator cannot have non-real eigenvalues, L_0 cannot be self-adjoint. In the terminology of Reed and Simon [12], the problem is not *quantum mechanically complete*.

The same argument can be made for any second-order Sturm–Liouville equation which has a two-dimensional space of square integrable solutions. To obtain a self-adjoint operator a single boundary condition $U(1) = 0$ is insufficient, just as in the case of a Sturm–Liouville equation with two regular endpoints: the domain of L_0 must be restricted by imposing an additional boundary condition at $r = 0$. If we fail to impose such an extra condition we get an operator which is not symmetric.

The procedure for setting such conditions is described in a number of sources: see, e.g. [4, p 246]. Observe that for $\lambda = 0$ equation (18) has the solutions

$$u_0(r) = \sin(\epsilon^{-1}\log(r)),\ v_0(r) = \cos(\epsilon^{-1}\log(r)). \tag{19}$$

Admissible self-adjoint boundary conditions have the form

$$A[U, u_0](0) + B[U, v_0](0) = 0, \tag{20}$$

where (A, B) is a nontrivial pair of real constants, and $[f, g](0)$ denotes the Wronskian limit

$$[f, g](0) = \lim_{r\searrow 0} r(f(r)g'(r) - f'(r)g(r)). \tag{21}$$

(Remark: the function $Au_0 + Bv_0$ is the function denoted by $\hat{\chi}_\infty$ in [4, p 246]; the particular formulation (20) appears in [11].)

From now on, for definiteness, we shall restrict L_0 by imposing the boundary condition

$$[U, u_0](0) = 0. \tag{22}$$

This condition may be regarded approximately as requiring that U either oscillate 'in phase' with u_0 near $r = 0$ or else tend to zero so rapidly that the phase difference between U and u_0 becomes irrelevant. We shall denote the resulting restriction of L_0 by L_0'; its domain is

$$\begin{aligned} D(L_0') &= \{U \in D(L_0) | [U, u_0](0) = 0\} \\ &= \left\{ U \in L_r^2(0,1) \left| \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}U}{\mathrm{d}r}\right) + \frac{U}{\epsilon^2 r^2} \in L_r^2(0,1);\ U(1) = 0 = [U, u_0](0) \right.\right\}. \end{aligned}$$

It is now an easy matter to check that $\lambda = 0$ is an eigenvalue of L_0' with eigenfunction u_0.

In order to describe the rest of the spectrum of L_0' we make one further observation. The equation (18) has the additional property that for every real λ, all solutions have infinitely many zeros in a neighbourhood of the origin, as may easily be verified using a Frobenius ansatz $U(r) \sim \Re(r^{\pm i/\epsilon})$ or $U(r) \sim \Im(r^{\pm i/\epsilon})$. Singular Sturm–Liouville problems for which every real solution of the equation is square integrable and has infinitely many zeros are called *limit circle oscillatory*. Such problems have the property that, no matter what self-adjoint boundary conditions may be imposed, the spectrum consists of eigenvalues tending to $-\infty$ and to $+\infty$, with no finite accumulation points. Proofs of these results may be found in, e.g., Weidmann [14, theorem 7.11; theorem 14.3; corollary 14.5a]; a review of theoretical results on their spectra (without proofs) together with software for their numerical solution may be found in [2].

It follows that the spectrum of L_0' consists of eigenvalues only and that these accumulate at $\pm\infty$ and at no finite point,

$$\cdots < \lambda_{0,-2} < \lambda_{0,-1} < 0 < \lambda_{0,1} < \lambda_{0,2} < \cdots. \tag{23}$$

A self-adjoint realization of our original Laplace operator L is thus given by

$$L' = L_0' \bigoplus_{n=1}^{\infty} L_n; \tag{24}$$

its spectrum is

$$\sigma(L') = \sigma(L'_0)\bigcup_{n=1}^{\infty}\sigma(L_n). \tag{25}$$

Thus the spectrum of the self-adjoint Laplace operator L' is purely discrete and consists of eigenvalues accumulating at $\pm\infty$ and at no finite point. There is no continuous spectrum.

The particular choice of the self-adjoint realization of the operator L'_0 does not affect the above conclusion. It is known that for ordinary differential operators of limit-circle type the qualitative properties of the spectrum are the same for all realizations.

As a final remark, we observe that the boundary condition (22) on elements u of the domain of L'_0 may also be written as

$$\int_0^1 u_0(r)[-r^{-1}(rU')' - \epsilon^{-2}r^{-2}U]r\,\mathrm{d}r = 0. \tag{26}$$

This translates into a boundary condition on elements of the operator L',

$$\int_{\Omega_1} u_0\Delta u\,\mathrm{d}x\,\mathrm{d}y = 0 \tag{27}$$

which elements of $D(L')$ must satisfy, in addition to (2), (3).

4. Eigenvalue asymptotics

The qualitative study of the spectrum presented above can be complemented by a quantitative description. For a self-adjoint operator L and $\lambda > 0$, we denote by $n_\pm(\lambda, L)$ the number of eigenvalues of $\pm L$ in the interval $(0, \lambda)$. It is known that for a majority of naturally stated problems the *distribution functions* $n_\pm(\lambda, L)$ have power asymptotics in λ as $\lambda \to \infty$; there is a huge literature on the topic and a (far from being complete) review of results in this field can be found in [13]. We will see that the asymptotics of $n_\pm(\lambda)$ for our problem are different for $\lambda \to +\infty$ and for $\lambda \to -\infty$: in a certain sense there are considerably fewer negative eigenvalues than positive ones.

The reasoning given below is based upon classical facts about the asymptotic behaviour of solutions of Sturm–Liouville equations that can be found, say, in [9, chapter 11], partly because this makes the analysis more general than an argument based on the explicit use of Bessel functions, and partly because the authors find this approach more accessible. We omit the tedious calculations as well as the justification of the asymptotic formulae, which are standard.

We represent our operator L' as a direct sum

$$L' = L'_0 \oplus \tilde{L}, \qquad \tilde{L} = \bigoplus_{n=1}^{\infty} L_n. \tag{28}$$

The spectrum of L' is thus the union of the spectra of L'_0 and $\tilde{L}$. We consider $\tilde{L}$ first. The eigenvalues of $\tilde{L}$ due to (18), are given by

$$\lambda_{n,k} = j_{2n,k}, \tag{29}$$

the zeros of the Bessel functions $J_{2n}(r), n = 1, 2, \ldots$. Note that these eigenvalues do not depend on the parameter ϵ in (3), in particular, we can set $\epsilon = 0$. In this case the boundary condition (3) takes the form $u = 0$ for $r \in (0, 1)$. Together with (2), this means that the eigenvalues of the operator $\tilde{L}$ are the same as the eigenvalues of the Laplacian in Ω_1 with

Dirichlet boundary conditions on the whole of the boundary of Ω_1. The eigenvalue asymptotics for this problem is well known, thus we have

$$n_+(\lambda, \tilde{L}) \sim \frac{\pi^2}{4}\lambda, \qquad \lambda \to +\infty. \tag{30}$$

Of course, there are no negative eigenvalues for $\tilde{L}$, therefore $n_-(\lambda, \tilde{L}) = 0$.

Now we consider the spectrum of the operator L_0' in (19). By the change $v(t) = t^{-\frac{1}{2}}u(t)$, equation (19) transforms to

$$-v_{rr} - \left(\frac{1}{4} + \frac{1}{\epsilon^2}\right) r^{-2} v = \lambda v, \tag{31}$$

on the interval (0, 1) with the boundary conditions

$$v(1) = 0, \qquad v(r) = a_0 \sin(\epsilon^{-1} \log r) + o(1), \qquad r \to 0, \tag{32}$$

the latter stemming from (23).

To study the eigenvalue asymptotics for the problem (31) we make a change of variables $t = \kappa r$, $w(t) = v(\kappa r)$, with $\kappa = |\lambda|^{\frac{1}{2}}$. This change transforms our eigenvalue problem to

$$-w_{tt} + \left(\frac{1}{4} + \frac{1}{\epsilon^2}\right) t^{-2} w = \sigma w, \qquad \sigma = \frac{\lambda}{|\lambda|}, \tag{33}$$

with boundary conditions (normalized by setting $a_0 = 1$)

$$\begin{aligned} &w(\kappa) = 0, \\ &w(t) = \cos(\epsilon^{-1}\log\kappa)\sin(\epsilon^{-1}\log t) + \sin(\epsilon^{-1}\log\kappa)\cos(\epsilon^{-1}\log t) + o(1), \qquad t \to 0. \end{aligned} \tag{34}$$

Thus the eigenvalues of the boundary problem (31) and (32) have the form $\sigma\kappa^2$, where κ is a positive zero of the function $w(t)$ that satisfies (33) with appropriate $\sigma = \pm 1$ and with the boundary condition as in (34) at $t = 0$.

We denote by $w_c(t)$, $w_s(t)$ the solutions of equation (33), having as $t \to 0$ asymptotics $w_c(t) \sim \cos(\epsilon^{-1}\log t)$, resp., $w_s(t) \sim \sin(\epsilon^{-1}\log t)$. So, our eigenfunction $w(t)$ must have the form

$$w(t) = \cos(\epsilon^{-1}\log\kappa) w_s(t) + \sin(\epsilon^{-1}\log\kappa) w_c(t). \tag{35}$$

Consider the case of $\sigma = 1$ first. Then equation (33) is oscillatory at infinity and therefore the solutions $w_{sc}(t)$ have for $t \to \infty$ asymptotics

$$w_s(t) \sim A_1 \sin t + B_1 \cos t; \qquad w_c(t) = A_2 \sin t + B_2 \cos t, \qquad t \to \infty, \tag{36}$$

with some real coefficients A_j, B_j. Therefore, large zeros of the function $w(t)$ are asymptotically described by the equation

$$w(\kappa) = \cos(\epsilon^{-1}\log\kappa)(A_1 \sin\kappa + B_1\cos\kappa) + \sin(\epsilon^{-1}\log\kappa)(A_2\sin\kappa + B_2\cos\kappa) = 0. \tag{37}$$

To study the behaviour of the roots of equation (37), we represent it in the form

$$\tan(\epsilon^{-1}\log\kappa) = -\frac{A_1\sin\kappa + B_1\cos\kappa}{A_2\sin\kappa + B_2\cos\kappa}. \tag{38}$$

Just by the observation of the graphs of the functions in (38), one can see that on each period $\pi n, \pi(n+1)$ of the function on the right-hand side, there are only finitely many, uniformly in n, roots of equation (38). Thus, the counting function $n_+(\lambda)$ of positive eigenvalues κ^2 of our problem below λ is estimated as $n_+(\lambda, L_0') = O\big(\lambda^{\frac{1}{2}}\big)$.

Now consider the negative eigenvalues, i.e., $\sigma = -1$. The equation (33) is non-oscillatory at infinity and therefore the solutions $w_{sc}(t)$ have at infinity the asymptotics

$$w_s(t) = A\exp(t)(1 + o(1)), \qquad w_c(t) = B\exp(t)(1 + o(1)), \qquad t \to \infty. \tag{39}$$

Therefore, to determine the asymptotics of the zeros of the function $w(t)$, we must consider the equation

$$A\cos(\epsilon^{-1}\log\kappa) + B\sin(\epsilon^{-1}\log\kappa) = 0. \tag{40}$$

Equation (40) can be written in the form $\sin(\theta_0 + \epsilon^{-1}\log\kappa) = 0$ where $\sin(\theta_0) = \frac{A}{\sqrt{A^2+B^2}}$, thus the zeros of $w(t)$ are described by

$$\kappa_n \sim C\exp\left(\frac{\pi n}{\epsilon}\right),$$

and therefore for the distribution function $n_-(\lambda)$, $\lambda > 0$, the number of negative eigenvalues above $-\lambda$, we have the asymptotics

$$n_-(\lambda, L_0') \sim \epsilon^{-1}\frac{\log\lambda}{2\pi}. \tag{41}$$

Taking into account (30) and (28), we arrive at the following conclusions. In the asymptotics of the positive eigenvalues, the contribution of L_0' is absorbed by that of $\tilde{L}$. So, up to lower order terms, asymptotically the positive spectrum of L' does not reflect the singularity in the boundary condition. On the other hand, the asymptotics of the negative eigenvalues of L' is determined by the boundary conditiononly. These eigenvalues are located extremely, and their density decreases as the parameter ϵ tends to zero.

Note, finally, that the asymptotics found above does not depend on the choice of the additional boundary condition at the singular point (the D-point) in the original problem: in other words, they do not depend on the choice of self-adjoint realization of L_0.

5. Other domains

Domains of different shape from that considered here, or with several zeros of f on the boundary, can be treated by repeated application of the Glazman decomposition trick (see [6]), which we now describe.

Let Ω_2 be another bounded regular domain, disjoint from Ω_1, with piecewise smooth boundary and no non-convex corners and whose boundary intersects the boundary of Ω_1 precisely on the semi-circular arc:

$$\Omega_1 \cap \Omega_2 = \emptyset, \qquad \partial\Omega_1 \cap \partial\Omega_2 = \{(\cos\theta, \sin\theta) \mid -\pi/2 \leqslant \theta \leqslant \pi/2\}. \tag{42}$$

Let $\Omega := \Omega_1 \cup \Omega_2$ and assume that $\partial\Omega$ has no non-convex corners. We consider in $L^2(\Omega)$ a Laplace operator T defined by removing the boundary condition $u = 0$ on the semicircular arc (which is not part of the boundary of $\Omega_1 \cup \Omega_2$) and imposing instead a regular self-adjoint boundary condition on $\partial\Omega\backslash\partial\Omega_1$. For simplicity of exposition we consider a Dirichlet condition:

$$u = 0 \text{ on } \partial\Omega\backslash\partial\Omega_1. \tag{43}$$

Once again, all boundary conditions should be understood in the weak sense. The domain of T will be

$$D(T) = \{u \in L^2(\Omega) \mid \Delta u \in L^2(\Omega),\ u \text{ satisfies (3), (27), (43)}\}. \tag{44}$$

We now compute the resolvent operator $(T - zI)^{-1}$ for non-real z: by showing that this is compact, we shall establish that T also has only discrete spectrum.

Let $K_1(z)$ and $K_2(z)$ be Poisson operators for the domains Ω_1 and Ω_2, respectively: this means that if $h \in H^s(\Gamma)$ for some $s \in \mathbb{R}$ then the functions

$$v_1 = K_1(z)h, \qquad v_2 = K_2(z)h$$

solve, in the appropriate weak sense [1], the following boundary value problems:

$$\left.\begin{array}{r}-\Delta v_1 - z v_1 = 0 \text{ in } \Omega_1,\\ v_1 \text{ satisfies (3), (27),}\\ v_1|_\Gamma = h;\end{array}\right\} \tag{45}$$

$$\left.\begin{array}{r}-\Delta v_2 - z v_2 = 0 \text{ in } \Omega_2,\\ v_2 = 0 \text{ satisfies (43),}\\ v_2|_\Gamma = h.\end{array}\right\}. \tag{46}$$

These operators are well defined for all non-real z. Let Δ_2 denote the usual Dirichlet Laplace operator in $L^2(\Omega_2)$:

$$D(\Delta_2) = H^2(\Omega_2) \cap H_0^1(\Omega_2). \tag{47}$$

Suppose that $u \in D(T)$ and $(T - zI)u = f \in L^2(\Omega)$. Let $f_1 = f|_{\Omega_1}$, $f_2 = f|_{\Omega_2}$, $u_1 = u|_{\Omega_1}$ and $u_2 = u|_{\Omega_2}$. Taking $h = u|_\Gamma$ we can decompose

$$u_1 = K_1(z)h + w_1, \qquad u_2 = K_2(z)h + w_2,$$

where $w_1 \in D(L')$, $w_2 \in D(\Delta_2)$, and

$$(L' - zI)w_1 = f_1, \qquad (\Delta_2 - zI)w_2 = f_2.$$

Thus $w_1 = (L' - zI)^{-1} f_1$ and $w_2 = (\Delta_2 - zI)^{-1} f_2$, giving

$$u_1 = K_1(z)h + (L' - zI)^{-1} f_1, \qquad u_2 = K_2(z)h + (\Delta_2 - zI)^{-1} f_2. \tag{48}$$

The function h is eliminated by insisting that the normal derivatives of u_1 and u_2 match across Γ:

$$\partial_1 u_1 + \partial_2 u_2 = 0, \tag{49}$$

where ∂_j denotes the outward normal derivative on Γ from Ω_j, $j = 1, 2$. Recall also that the normal derivatives of $K_1(z)h$ and $K_2(z)h$ are given by the Dirichlet to Neumann maps associated with L' and Δ_2, respectively:

$$\partial_1 K_1(z)h = \Lambda_1(z)h, \qquad \partial_2 K_2(z)h = \Lambda_2(z)h. \tag{50}$$

Combining (48), (49) and (50) to eliminate h results in the expression

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = R(z) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

where $R(z)$ is the block operator matrix representation of $(T - zI)^{-1}$ given by

$$\begin{pmatrix} [I - K_1(z)(\Lambda_1(z) + \Lambda_2(z))^{-1}\partial_1](L' - zI)^{-1} & -K_1(z)(\Lambda_1(z) + \Lambda_2(z))^{-1}\partial_2(\Delta_2 - zI)^{-1} \\ -K_2(z)(\Lambda_1(z) + \Lambda_2(z))^{-1}\partial_1(L' - zI)^{-1} & [I - K_2(z)(\Lambda_1(z) + \Lambda_2(z))^{-1}\partial_2](\Delta_2 - zI)^{-1} \end{pmatrix}. \tag{51}$$

We have already shown that L' has eigenvalues accumulating only at infinity and no essential spectrum. This means that $(L' - zI)^{-1}$ is compact (and self-adjoint, if z is real and not one of the eigenvalues of L'). The resolvent of the classical Dirichlet Laplacian Δ_2 is also well known to be compact [12, vol IV] since it has only eigenvalues accumulating only at infinity, and no essential spectrum. To show that $R(z)$ is a compact operator it is therefore sufficient to know that the following operators are all bounded in the appropriate L^2 spaces:

$$K_1(z); \quad K_2(z); \quad (\Lambda_1(z) + \Lambda_2(z))^{-1}\partial_1; \quad (\Lambda_1(z) + \Lambda_2(z))^{-1}\partial_2. \tag{52}$$

The first two operators here are Poisson operators and are known to be bounded from $H^s(\Gamma)$ to $L^2(\Omega_1)$ and $L^2(\Omega_2)$ respectively, for any $s \geqslant -1/2$—see [8, chapter 3].

For the remaining two operators we note that $\Lambda_1(z) + \Lambda_2(z)$ is a pseudodifferential operator of order 1 on the scale of Sobolev spaces $H^s(\Gamma)$ (see, e.g. [7, 10]) and that it fails to be invertible precisely when it has a non-trivial kernel. When this happens, we can take h to be an element of this kernel and deduce that the function w given by

$$w|_{\Omega_1} = K_1(z)h, \qquad w|_{\Omega_2} = K_2(z)h$$

is an eigenfunction of T with eigenvalue z. Since T is self-adjoint and $\Im(z) \neq 0$ this is impossible. The inverse $(\Lambda_1(z) + \Lambda_2(z))^{-1}$ therefore exists; it is a pseudodifferential operator of order -1 on the scale of Sobolev spaces $H^s(\Gamma)$. Since ∂_1 and ∂_2 are pseudodifferential operators of order 1, the operators

$$(\Lambda_1(z) + \Lambda_2(z))^{-1}\partial_1; \qquad (\Lambda_1(z) + \Lambda_2(z))^{-1}\partial_2$$

are bounded on the scale of Sobolev spaces $H^s(\Gamma)$.

Thus $R(z)$ is compact, so $(T - zI)^{-1}$ is compact, and our T has a purely discrete spectrum accumulating at $\pm\infty$, just like L'.

6. Discussion

Now we can return to the unusual result obtained in [3]. In the study of the spectrum, the authors did not consider the self-adjointness question. As a result they did not put the extra boundary condition at the singular point, a condition necessary for self-adjointness. So, the eigenvalue problem was being solved for an operator which was not only non-self-adjoint, but even nonsymmetric. As a result each real number turned out to be an eigenvalue (in fact, each complex number turns out to be an eigenvalue as well). Since each eigenfunction found in [3] satisfies *some* boundary conditions of the type (20), such an eigenfunction is an eigenfunction of *some* self-adjoint realization of the operator. So, the 'spectrum' found in [3] is just the union of the spectra of all possible self-adjoint realizations of the operator, and therefore there is nothing strange now that it covers the whole real line. It remains unclear what is the physical meaning of the particular choice of self-adjoint realization in such singular situations.

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